FREE VIBRATIONS OF A CIRCULAR RING WITH EQUIDISTANT MASSES

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The problem considered is the free vibrations of a thin rod of constant cross-section with centerline describing a circle and which carries equal and equidistant masses. It is assumed that the vibrations of the rod consist of flexural vibrations in the plane of its centerline and vibrations accompanied by displacements perpendicular to the plane of the axis and by torsion. A solution is obtained for both types of vibration in the case where it is possible to consider that the centerline of the rod is inextensible, the portions of the ring between the masses are inertialess and the rotational inertia of the masses is neglected. The problem is of interest for the investigation of vibration in a number of machine elements, in particular, the frames of certain types of centrifugal mills, the stators of electrical machines, turbine disks with massive rings [1], and so forth. An approximate solution for vibrations perpendicular to the plane of the ring was given in [1].

1. With the above-mentioned assumptions, in the case of flexural vibrations in the plane of the ring, the tangential displacement $v(\theta)$ of a point on the centerline and its first three derivatives are continuous, but the fourth and fifth derivatives have discontinuities at the points where the masses are located, i.e. at $\theta = k\alpha$; the magnitudes of the jumps in the derivatives are found from the well-known relations between the shear forces acting on the cross-section of the rod and the displacements of the points on its centerline

$$\frac{d^4}{d\theta^4} v(k\alpha+0) - \frac{d^4}{d\theta^4} v(k\alpha-0) = \frac{M\lambda^2R^3}{EI}\frac{d}{d\theta} v(k\alpha)$$

$$\frac{d^5}{d\theta^5} v(k\alpha+0) - \frac{d^5}{d\theta^5} v(k\alpha-0) = -\frac{M\lambda^2R^3}{EI} v(k\alpha)$$

$$(k = 0, 1, \ldots, n-1) \tag{1.1}$$

Here k is the number of the mass, θ is the central angle between the current point on the centerline and the mass with number zero, $\alpha = 2\pi/n$ is the central angle between successive masses, n is the number of masses, λ is the frequency of vibration, M is the magnitude of the attached mass, EI is the corresponding flexural rigidity, and R is the radius of the circle formed by the centerline.

The segments of the ring between the masses are assumed to be inertia-less, hence for $(k-1)\alpha < \theta < k\alpha$ $(k = 0, \ldots, n-1)$ the displacement $v(\theta)$ satisfies the equation [2]

$$\frac{d^6 v}{d\theta^6} + 2\frac{d^4 v}{d\theta^4} + \frac{d^2 v}{d\theta^2} = 0 \tag{1.2}$$

The general integral of this equation in the interval $(k-1)\alpha \leqslant \theta \leqslant k\alpha$ we take in the form

$$v(\theta) = \sum_{r=0}^{5} v_k^{(r)} \Phi_r(\theta - k\alpha) \tag{1.3}$$

where

$$v_k^{(0)} = v(k\alpha), \qquad v_k^{(r)} = \frac{d^r}{d\theta^r} v(k\alpha - 0) \qquad (r = 1, \ldots, 5) \tag{1.4}$$

$$\Phi_0(\theta) = 1 \qquad \Phi_1(\theta) = \theta$$

$$\Phi_2(\theta) = 2 - \frac{1}{2}\theta\sin\theta - 2\cos\theta, \qquad \Phi_3(\theta) = 2\theta + \frac{1}{2}\theta\cos\theta - \frac{5}{2}\sin\theta$$

$$\Phi_4(\theta) = 1 - \frac{1}{2}\theta\sin\theta - \cos\theta, \qquad \Phi_5(\theta) = \theta + \frac{1}{2}\theta\cos\theta - \frac{3}{2}\sin\theta \tag{1.5}$$

The functions $\Phi_r(\theta)$ form a fundamental system of solutions of equation (1.2) with a unitary matrix.

After differentiating the integral (1.3), substituting $\theta = (k-1)\alpha$ into it and replacing the values of the fourth and fifth derivatives of $v(\theta)$ after the $(k-1)$th mass by their values before this mass in accordance with the conditions (1.1), we obtain the totality of recurrence formulas

$$v_{k-1}^{(r)} = \sum_{s=0}^{5} v_k^{(s)} \Phi_s^{(r)}(-\alpha), \qquad (r = 0, 1, 2, 3)$$

$$v_{k-1}^{(4)} = \sum_{s=0}^{5} v_k^{(s)} \Phi_s^{(4)}(-\alpha) - \beta v_{k-1}^{(1)}$$

$$v_{k-1}^{(5)} = \sum_{s=0}^{5} v_k^{(s)} \Phi_s^{(5)}(-\alpha) + \beta v_{k-1}^{(0)} \tag{1.6}$$

where

$$\beta = \frac{MR^3\lambda^2}{EI}, \qquad \Phi_s^{(r)}(\alpha) = \frac{d^r}{d\alpha^r}\Phi_s(\alpha) \qquad (r, s = 0, \ldots, 5).$$

The formulas (1.6) are a system of six linear homogeneous equations of first order in the finite differences of the six functions $v_k^{(r)}$ of a discrete argument of the number k of the mass.

For solution of the system (1.6) it is expedient to eliminate from it the variables $v_k^{(3)}$, $v_k^{(4)}$ and $v_k^{(5)}$; we determine them by the displacements $v(\theta)$ and their first two derivatives at the $(k-1)$th, kth and $(k+1)$th junction points; this is accomplished by use of the first three equations of system (1.6) and their counterparts for the kth and $(k+1)$th junction points

$$v_{k+1}^{(r)} = \sum_{s=1}^{5} v_k^{(s)}\Phi_s^{(r)}(\alpha) + \beta\left[\Phi_4^{(r)}(\alpha)\, v_k^{(1)} - \Phi_5^{(r)}(\alpha)\, v_k^{(0)}\right] \qquad (r = 0, 1, 2) \qquad (1.7)$$

As a result we obtain

$$v_k^{(3)} = \frac{1}{2(\Phi_3\Phi_5'' - \Phi_3''\Phi_5)}\{\Phi_5''\left[(v_{k+1}^{(0)} - v_{k-1}^{(0)} - 2\Phi_1 v_k^{(1)}) - \beta(\Phi_4 v_k^{(1)} - \Phi_5 v_k^{(0)})\right] - \Phi_5\left[(v_{k+1}^{(2)} - v_{k-1}^{(2)}) - \beta(\Phi_4'' v_k^{(1)} - \Phi_5'' v_k^{(0)})\right]\}$$

$$v_k^{(4)} = \frac{1}{2\Phi_4''}\left[v_{k+1}^{(2)} - 2\Phi_2'' v_k^{(2)} + v_{k-1}^{(2)} - \beta(\Phi_4'' v_k^{(1)} - \Phi_5'' v_k^{(0)})\right]$$

$$v_k^{(5)} = \frac{1}{2(\Phi_3\Phi_5'' - \Phi_3''\Phi_5)}\{-\Phi_3''\left[(v_{k+1}^{(0)} - v_{k-1}^{(0)} - 2\Phi_1 v_k^{(1)}) - \beta(\Phi_4 v_k^{(1)} - \Phi_5 v_k^{(0)})\right] + \Phi_3\left[v_{k+1}^{(2)} - v_{k-1}^{(2)} - \beta(\Phi_4'' v_k^{(1)} - \Phi_5'' v_k^{(0)})\right]\} \qquad (1.8)$$

Here and in the following, the argument of all functions $\Phi_r (r = 0, \ldots, 5)$ will be the angle α, and the prime denotes differentiation with respect to α.

The same three equations of the system (1.6) together with relations (1.7) and (1.8) lead to still another group of recurrence formulas which connect the tangential displacement and its first two derivatives at three adjacent junction points of the rod

$$\Phi_4''(v_{k+1}^{(0)} - 2v_k^{(0)} + v_{k-1}^{(0)}) + \beta(\Phi_4''\Phi_5 - \Phi_4\Phi_5'')\, v_k^{(0)} - \Phi_4(v_{k+1}^{(2)} + v_{k-1}^{(2)}) + 2(\Phi_2''\Phi_4 - \Phi_2\Phi_4'')\, v_k^{(2)} = 0$$

$$\beta(\Phi_4''\Phi_5' - \Phi_4'\Phi_5'')\, v_k^{(0)} + \Phi_4''(v_{k+1}^{(1)} - v_{k-1}^{(1)}) - \Phi_4'(v_{k+1}^{(2)} + v_{k-1}^{(2)}) + 2(\Phi_2''\Phi_4' - \Phi_2'\Phi_4'')\, v_k^{(2)} = 0$$

$$(\Phi_3''\Phi_5' - \Phi_3'\Phi_5'')(v_{k+1}^{(0)} - v_{k-1}^{(0)}) + (\Phi_3\Phi_5'' - \Phi_3''\Phi_5)(v_{k+1}^{(1)} - 2v_k^{(1)} + v_{k-1}^{(1)}) + 2\Phi_1(\Phi_3'\Phi_5'' - \Phi_3''\Phi_5')\, v_k^{(1)} - \beta\left[\Phi_4(\Phi_3''\Phi_5' - \Phi_3'\Phi_5'') + \Phi_4'(\Phi_3\Phi_5'' - \Phi_3''\Phi_5) + \Phi_4''(\Phi_3'\Phi_5 - \Phi_3\Phi_5')\right] v_k^{(1)} + (\Phi_3'\Phi_5 - \Phi_3\Phi_5')(v_{k+1}^{(2)} - v_{k-1}^{(2)}) = 0 \qquad (1.9)$$

The system (1.9) of three finite-difference equations of second order in the quantities $v_k^{(0)}$, $v_k^{(1)}$ and $v_k^{(2)}$ is equivalent to the system (1.6).

After sufficient cumbersome manipulations and reduction on the factor

$$^1/_2\,\alpha \sin\alpha\,(^1/_4\,\alpha^3 - {}^1/_4\,\alpha \sin^2\alpha + \alpha\cos\alpha - \alpha + \sin\alpha - \sin\alpha\cos\alpha)$$

by which the characteristic polynomials of the systems (1.6) and (1.9) are distinguished, the characteristic equation of the system (1.9) may be put in the form

$$\begin{gathered}e^{6q} + 1 - [4\cos\alpha + 2 - \beta\,(\alpha + \alpha\cos\alpha - 2\sin\alpha)]\,(e^{5q} + e^q) + \\ + [4\cos^2\alpha + 8\cos\alpha + 3 + \beta\,(4\sin\alpha + 4\sin\alpha\cos\alpha - 2\alpha - 6\alpha\cos\alpha) + \\ + \beta^2\,(^1/_4\,\alpha^2 + {}^1/_2\,\alpha^2\cos\alpha - {}^3/_2\,\alpha\sin\alpha - 2\cos\alpha + {}^1/_4\cos^2\alpha + {}^7/_4)]\,(e^{4q} + e^{2q}) - \\ - [8\cos^2\alpha + 8\cos\alpha + 4 - \beta\,(6\alpha + 2\alpha\cos\alpha + 4\alpha\cos^2\alpha - 4\sin\alpha - 8\sin\alpha\cos\alpha) - \\ - \beta^2(2\alpha\sin\alpha + \alpha\sin\alpha\cos\alpha + 3\cos^2\alpha - {}^1/_2\sin^2\alpha - 4\cos\alpha - {}^3/_2\alpha^2 + 1)]\,e^{3q} = 0 \qquad (1.10)\end{gathered}$$

From the condition of periodicity of the solution of the difference equations

$$v_0^{(r)} = v_n^{(r)} \qquad (r = 0, 1, \ldots, 5) \qquad (1.11)$$

it follows that the characteristic equation (1.10) must have roots of the form

$$e^q = \exp\frac{2\pi mi}{n} = e^{\alpha mi} \qquad (i^2 = -1,\ m \text{ is any integer}) \qquad (1.12)$$

Substitution of (1.12) into (1.10) leads to a quadratic equation for the calculation of all the frequencies of free vibration for any number of masses

$$\begin{gathered}\beta^2[2\cos m\alpha\,(^1/_4\,\alpha^2 + {}^1/_2\,\alpha^2\cos\alpha - {}^3/_2\,\alpha\sin\alpha - 2\cos\alpha + {}^1/_4\cos^2\alpha + {}^7/_4) - \\ - {}^3/_2\,\alpha^2 + 2\alpha\sin\alpha + \alpha\sin\alpha\cos\alpha - 4\cos\alpha + 3\cos^2\alpha - {}^1/_2\sin^2\alpha + 1] + \\ + \beta\,[2\cos 2m\alpha\,(\alpha + \alpha\cos\alpha - 2\sin\alpha) + 2\cos m\alpha\,(4\sin\alpha\cos\alpha + \\ + 4\sin\alpha - 6\alpha\cos\alpha - 2\alpha) + 6\alpha + 2\alpha\cos\alpha + 4\alpha\cos^2\alpha - 4\sin\alpha - \\ 8\sin\alpha\cos\alpha] + [2\cos 3m\alpha - 2\cos 2m\alpha\,(4\cos\alpha + 2) + 2\cos m\alpha\,(4\cos^2\alpha + 8\cos\alpha + 3) - \\ - 8\cos^2\alpha - 8\cos\alpha - 4] = 0 \qquad (1.13)\end{gathered}$$

It should be noted that in view of the high degree of symmetry of the system, the number of different values of the frequency determined from the frequency equation (1.13) is less than $2n$ - the number of its degrees of freedom; moreover, one and the same frequency corresponds to two linearly independent forms of the tangential displacement of the mass

$$(v_{km}^{(0)})_1 = \cos mk\alpha, \qquad (v_{km}^{(0)})_2 = \sin mk\alpha \tag{1.14}$$

The corresponding forms of the radial displacement of the mass $v_{km}^{(1)}$, and also the shapes of the rod may be constructed in accordance with relations (1.9), (1.8), (1.3) and (1.1).

Expansion of the frequency equation (1.13) in a power series in α

$$\left(\frac{\rho R^4\lambda^2}{EI}\right)^2 (0\cdot\alpha^6 + \cdots) + \frac{\rho R^4\lambda^2}{EI}[(m^2+1)\alpha^6 + \cdots] - [m^2(m^2-1)^2\alpha^6 + \cdots] = 0$$
$$(\beta = nM/2\pi R) \tag{1.15}$$

shows that if the number of masses is increased without bound, while the sum of the masses remains a constant, then one root of the frequency equation for our system approaches the value of the corresponding frequency of vibration of a ring of density ρ, whereas the other root increases without bound; this unbounded growth is explained by the fact that the condition of inextensibility of the centerline of the ring with an unlimited increase in the number of masses is equivalent to the imposition of some absolutely rigid constraint. It is important that the frequency of the latter type may not be found by such an approximate method as was used, for example, in [1].

2. We consider vibration accompanied by displacements perpendicular to the plane of the centerline of the rod and by torsion.

The deflection $w(\theta)$ for vibrations of this type satisfies the differential equation containing delta functions $\delta(\theta)$

$$\frac{d^6w}{d\theta^6} + 2\frac{d^4w}{d\theta^4} + \frac{d^2w}{d\theta^2} = \sum_{k=0}^{n-1}\left[\frac{MR^3\lambda^2}{GI}\delta(\theta - k\alpha) - \frac{MR^3\lambda^2}{EI_1}\frac{d^2}{d\theta^2}\delta(\theta - k\alpha)\right]w(k\alpha) \tag{2.1}$$

Here EI_1 is the corresponding flexural rigidity, and GI is the torsional rigidity of the rod.

The general integral of equation (2.1) for $\theta = k\alpha$ reduces to a recurrence formula for determination of the deflection $w(k\alpha)$ of a point at a mass

$$w(k\alpha) = \sum_{r=0}^{5}\frac{d^r w(-0)}{d\theta^r}\Phi_r(k\alpha) + \beta\sum_{j=0}^{k}\{\Phi_3[(k-j)\alpha] - \gamma\Phi_5[(k-j)\alpha]\}w(j\alpha) \tag{2.2}$$

where

$$\beta = \frac{MR^3\lambda^2}{EI_1}, \qquad \gamma = \frac{EI_1}{GI}$$

Formula (2.2) is equivalent to the corresponding finite-difference equations and also allows the determination of the deflection of the *k*th mass as a function of its number. If the function

$$W(z)=\sum_{k=0}^{\infty} w(k\alpha)\, z^k \tag{2.3}$$

is known, then for the calculation of the quantity $w(k\alpha)$ it is sufficient to know the derivatives of $W(z)$ for $z = 0$. After multiplication by z^k and summation, formula (2.2) gives

$$\{1-\beta\,[\Phi_3^*(z)-\gamma\Phi_5^*(z)]\}\, W(z)=\sum_{r=0}^{5}\frac{d^r w(-0)}{d\theta^r}\,\Phi_r^*(z) \tag{2.4}$$

where

$$\Phi_r^*(z)=\sum_{k=0}^{\infty}\Phi_r(k\alpha)\, z^k \qquad (r=0,1\,\ldots,5). \tag{2.5}$$

The function $W(z)$ defined by (2.4) is a rational fractional function and may be represented in the form

$$W(z)=\frac{1}{Q(z,\beta)}\sum_{r=0}^{5}\frac{d^r w(-0)}{d\theta^r}\,P_r(z) \tag{2.6}$$

where $P_r(z)$ and $Q(z, \beta)$ are polynomials in z, and $W(z)$ is regular at infinity; the values of its derivatives for $z = 0$ may be determined by the corresponding expansion according to the roots of the polynomial $Q(z, \beta)$.

From the condition of periodicity

$$w(0)=w(n\alpha) \tag{2.7}$$

it follows that the polynomial $Q(z, \beta)$ must have roots of the form

$$z=\exp\frac{2\pi mi}{n}=e^{\alpha mi} \tag{2.8}$$

The equality

$$Q(e^{\alpha mi},\beta)=0 \tag{2.9}$$

after certain transformations reduces to an explicit expression for the frequency of free vibration

$$\lambda^2=\frac{EJ_1}{MR^3}\,4(\cos m\alpha-1)(\cos m\alpha-\cos\alpha)^2:[2(2-\gamma)\,\alpha(\cos m\alpha-\cos\alpha)^2+$$
$$+(1-\gamma)\,\alpha(\cos m\alpha-1)(\cos\alpha\cos m\alpha-1)-(5-3\gamma)\sin\alpha(\cos m\alpha-1)(\cos m\alpha-\cos\alpha)] \tag{2.10}$$

For vibrations of the type considered, as well as in the case of vibrations in the plane of the centerline of the rod, one frequency serves for two linearly independent modes of vibration of the form (1.14); the number of values of the frequency determined from (2.10) for different m is as before less than n – the number of degrees of freedom of the system.

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